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Gauge freedom of Dirac theory in complexified spacetime algebra

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Abstract

A Dirac theory over complexified spacetime algebra leads unavoidably to a $U(1) \otimes U(1) \otimes SU(3)$ global gauge freedom. We argue the $SU(3)$ freedom corresponds to colour. The $U(1) \otimes U(1)$ gauge freedom mixes to give two photon fields responsible for the charge assignments of the standard model. In addition to the usual up/down spin and particle/anti-particle decomposition of the Dirac field, this 16-dimensional field decomposes into a colourless leptonic component and a three-coloured quarkonic component.

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1. Introduction

The notion of $SU(3)$ colour was originally introduced as a book keeping device by Gell-Mann to restore the Fermi–Dirac statistics of quarks. Since then it has gained respectability as a gauge theory called quantum chromodynamics (QCD). However, the $SU(3)$ gauge appears to have no fundamental physical explanation, or does it? We demonstrate this in a complexified spacetime algebra formulation of the Dirac theory where the $SU(3)$ gauge is present. An analysis of the gauge in this context reveals that the Dirac field is composed of a colourless leptonic component and a three-coloured quark component. Further, the maximal gauge freedom is given by the group $U(1) \otimes U(1) \otimes SU(3)$. A $U(1)$ phase freedom together with another $U(1)$ non-phase freedom. The associated $U(1)$ gauge fields mix to give the correct charge assignments for the particles within each generation of the standard model. Moreover, the approximate flavour symmetries play no role whatsoever and are not needed, although they are useful as an analytical device.

Dirac theory originates from the early work of Dirac on relativistic electron theory. His theory, which is now the established one [1, 2], is based on a 4×4 matrix representation of the Clifford algebra $\mathbb{C} \otimes C\ell(1, 3)$. The Dirac gamma matrices provide a representation of a Hilbert space for particle/anti-particle and spin up/down fibred over Minkowski spacetime. The spacetime algebra approach, a programme revitalized by Hestenes [3, 4] in the sixties, avoids an abstract space of states and endeavours to interpret the *gamma matrices* as vectors in an associative algebra generated by a 1 + 3 orthonormal frame in Minkowski spacetime. In addition, Hestenes argues for the elimination of complex numbers and has developed a *real* Dirac theory [5] to that end. This paper adopts a similar spacetime algebra programme but not a real Dirac theory. A good modern introduction to Clifford algebra in this context is Lounesto [6].

The Dirac algebra $\mathbb{C}\otimes C\ell(1,3)$ is already known to permit $SU(3)$ gauge symmetries. The tetrahedral structure of idempotents resulting in $SU(3)$ symmetries was studied by Chisholm [7] and again in the context of quantum logic by Schmeikal [8, 9]. What is shown in this paper is that the residual freedom, once the Lorentz group is taken into account, provides an unavoidable $U(1)\otimes U(1)\otimes SU(3)$ gauge freedom. This gauge freedom is a result of spacetime covariance of the Dirac equation over complexified spacetime algebra. Chisholm and Farwell [9–12] have developed a unified spin model of gravitation and the standard model in the real Clifford algebra $C\ell(3,8)$. This formalism contains the standard model over $C\ell(1,6)$. Recently Trayling and Baylis [13] have formulated the standard model over $C\ell(7,0)$.

We consider the Dirac equation in the complexified spacetime algebra $\mathbb{C}\otimes C\ell(\eta)$, with vector generators $\mathbf{e}_\mu$ satisfying $\mathbf{e}_0^2=-\eta$ and $\mathbf{e}_k^2=\eta$, where $\eta=\pm1$ corresponds to the Lorentz/anti-Lorentz metric given by

$$\mathrm{i}\nabla\psi=m\psi\mathbf{e}_0. \tag{1}$$

This formulation, normally on the even subalgebra $\mathbb{C}\otimes C\ell^{+}(\eta)$, is the covariant version of Joyce [14] where the notation of this paper is established. This is equivalent to the classical Dirac equation on the left ideal $\mathbb{C}\otimes C\ell^{+}(\eta)\frac{1}{2}(1+\mathrm{i}\mathbf{e}_{12})$. See Joyce and Martin [15] for the details of this relationship. The complexified spacetime field ψ is composed of real multivectors multiplied by a phase. The real multivectors have a geometric interpretation (in the sense of Hestenes). The phase may be interpreted as the *quantum* component of ψ. An alternative perspective, and perhaps that which is in vogue, is to follow Kaluza–Klein theory. One introduces a compact fourth spatial dimension $\mathbf{e}_4$. Then $\mathbb{C}\otimes C\ell(\eta)\cong C\ell(1,4)$ with the unit imaginary i identified as $\mathbf{e}_{01234}$. We prefer the former view although the distinction is irrelevant in what follows.

The action of the Lorentz group is generated by the real Lie subalgebra $\langle\mathbf{e}_{0k},\mathbf{e}_{kl}\rangle_{\mathbb{R}}$. The Lie bracket is given by the anti-symmetric product $[\mathbf{a},\mathbf{b}]=\frac{1}{2}(\mathbf{ab}-\mathbf{ba})$. A 4-vector $\mathbf{x}$ transforms under a Lorentz transformation according to

$$\mathbf{x}\mapsto\mathbf{U}\mathbf{x}\mathbf{U}^{-1} \tag{2}$$

where

$$\mathbf{U}=\mathrm{e}^{\mathbf{m}\frac{\phi}{2}}\,\mathrm{e}^{\mathbf{n}\frac{\theta}{2}} \tag{3}$$

with $\mathbf{m}=m^k\mathbf{e}_{0k}$, $\mathbf{m}^2=1$, $\mathbf{n}=n^{kl}\mathbf{e}_{kl}$, $n^{lk}=-n^{kl}$ and $\mathbf{n}^2=-1$. This defines a rotation by θ in the plane given by $\mathbf{n}$ followed by a boast along $m^k\mathbf{e}_k$ with rapidity ϕ. The $\nabla=\mathbf{e}_\mu\partial^\mu$ operator transforms according to $\nabla\mapsto\mathbf{U}^{-1}\nabla\mathbf{U}$. The spinor ψ preserves the Dirac equation by transforming according to

$$\psi\mapsto\mathbf{U}^{-1}\psi. \tag{4}$$

An infinitesimal analysis in Joyce [14] shows that this left action implies that ψ is a spin half field. This leads to observables for up/down spin. These together with particle/anti-particle observables account for two of the four degrees of freedom. There are a total of four degrees (and not five) because the unit imaginary commutes with all multivectors. Moreover, the action under left or right multiplication is faithfully generated by only 16 generators. Six of these are required to generate the Lorentz group. This leaves ten which we chose in such a way as to generate the gauge group $U(1) \otimes U(1) \otimes SU(3)$. In principle, a larger gauge group could be chosen, however, this would be subject to constraints which when taken to account would reduce to $U(1) \otimes U(1) \otimes SU(3)$.

2. The master gauge group

We seek the largest (real) compact gauge group preserving solutions to the full Dirac equation

$$\mathrm{i}\nabla\psi = m\psi\mathbf{e}_0 \tag{5}$$

where $\psi \in \mathbb{C} \otimes C\ell(\eta)$ that preserves the *observable content* of the solutions. That is to say, the generators of the gauge group commute with the scalar energy operator $\mathrm{i}\partial^0$, the kth component scalar momentum operators $-\mathrm{i}\partial^k$ and the bivector spin operators $\frac{\mathrm{i}}{2}\mathbf{e}_{kl}$. First note that under the transformation $\psi \mapsto \mathbf{e}_{0123}\psi$ the observable content of ψ is preserved but now satisfies the equivalent equation

$$\mathrm{i}\nabla\psi = -m\psi\mathbf{e}_0 \tag{6}$$

having a negative mass. Thus under a right action a basis multivector $\mathbf{u}$ either commutes with $\mathbf{e}_0$ preserving the sign of the mass or anti-commutes reversing the sign of the mass term. In the latter case left multiplication by $\mathbf{e}_{0123}$ is required to prevent sign reversing. This leads to the following set of 16 (right acting) compact generators[1]:

$$\psi T_\emptyset = \mathrm{i}\psi \tag{7}$$
$$\psi T_0 = \sqrt{\eta}\psi\mathbf{e}_0 \tag{8}$$
$$\psi T_k = \sqrt{\eta}\mathbf{e}_{0123}\psi\mathbf{e}_k \tag{9}$$
$$\psi T_{0k} = \eta\mathbf{e}_{0123}\psi\mathbf{e}_{0k} \tag{10}$$
$$\psi T_{kl} = -\eta\psi\mathbf{e}_{kl} \tag{11}$$
$$\psi T_{0kl} = \sqrt{-\eta}\psi\mathbf{e}_{0kl} \tag{12}$$
$$\psi T_{123} = \sqrt{-\eta}\mathbf{e}_{0123}\psi\mathbf{e}_{123} \tag{13}$$
$$\psi T_{0123} = \mathrm{i}\mathbf{e}_{0123}\psi\mathbf{e}_{0123} \tag{14}$$

where $k, l \in \{1, 2, 3\}$ such that $k \neq l$. Let I be the set of words formed from the letters $\{0, 1, 2, 3\}$ composed of distinct letters. We denote the generators above by T_α where $\alpha \in I$ and the properties $T_\alpha^2 = -1$ and $T_{\pi\alpha} = (\operatorname{sgn}\pi)T_\alpha$ where $\pi \in S_{|\alpha|}$ hold. The generators satisfy the relations

$$T_\mu T_\nu = T_{\mu\nu} \tag{15}$$
$$T_\mu T_\nu T_\rho = \mathrm{i}T_{\mu\nu\rho} \tag{16}$$
$$T_0T_1T_2T_3 = \mathrm{i}T_{0123} \tag{17}$$

where $\mu, \nu, \rho \in \{0, 1, 2, 3\}$ are distinct. The commutation relations are now easily calculated to be

$$[T_0, T_k] = T_{0k} \qquad [T_{0k}, T_{0lm}] = T_{klm} \tag{18}$$
$$[T_{0k}, T_{0l}] = T_{kl} \qquad [T_{0kl}, T_{klm}] = -T_{0m} \tag{19}$$
$$[T_{0k}, T_{kl}] = -T_{0l} \qquad [T_{0kl}, T_{0lm}] = T_{km} \tag{20}$$

[1] The full spacetime version of the classical Dirac equation $\mathrm{i}\nabla\psi = m\psi$ has generators where the $\mathbf{e}_{0123}$ appearing on the left are replaced by the complex number i.

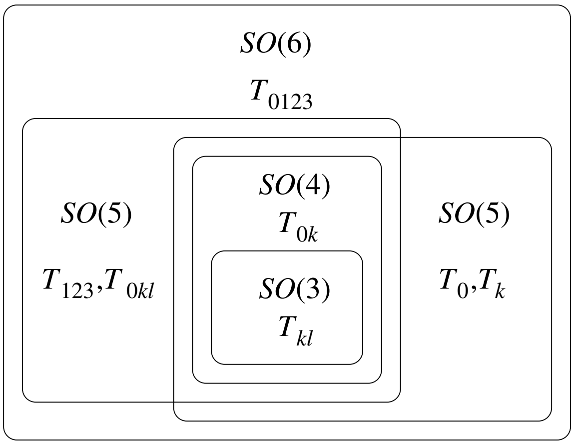

Figure 1. Important subgroups of the master gauge group.

$$[T_{kl}, T_{lm}] = -T_{km} \qquad [T_{klm}, T_{0k}] = T_{0lm} \tag{21}$$
$$[T_k, T_l] = T_{kl} \qquad [T_{0123}, T_0] = T_{123} \tag{22}$$
$$[T_{0kl}, T_m] = T_{0klm} \qquad [T_k, T_{0klm}] = T_{0lm} \tag{23}$$
$$[T_{0k}, T_k] = -T_0 \qquad [T_{kl}, T_l] = -T_k \tag{24}$$
$$[T_0, T_{123}] = T_{0123} \qquad [T_{0klm}, T_{0kl}] = -T_m \tag{25}$$
$$[T_{123}, T_{0123}] = T_0 \qquad [T_0, T_{0k}] = -T_k \tag{26}$$

where k, l, m are distinct. All other Lie brackets are zero.

The first generator $T_\emptyset$ commutes with the other generators and generates $U(1)$. The remaining 15 generate $SO(6)$. Moreover, the adjoint matrix representation is given by interpreting

$$\begin{bmatrix} 0 & T_0 & T_1 & T_2 & T_3 & T_{0123} \\ -T_0 & 0 & -T_{01} & -T_{02} & -T_{03} & T_{123} \\ -T_1 & -T_{01} & 0 & -T_{12} & T_{31} & -T_{023} \\ -T_2 & T_{02} & T_{12} & 0 & -T_{23} & -T_{031} \\ -T_3 & T_{03} & -T_{31} & T_{23} & 0 & -T_{012} \\ -T_{0123} & -T_{123} & T_{023} & T_{031} & T_{012} & 0 \end{bmatrix} \tag{27}$$

as follows. The matrix generator corresponding to T_α is given by replacing all its appearances by 1 with all other entries 0. For example, $(T_{01})_{ab}$ has all components zero except for $(T_{01})_{32} = 1$ and $(T_{01})_{23} = -1$. These matrices satisfy $TT^t = I$ and $T + T^t = 0$.

The master gauge group has important subgroups as given in figure 1. On $\mathbb{C} \otimes C\ell^+(\eta)$ the Dirac equation is invariant under $U(1) \otimes U(1) \otimes SO(4)$. However, the spin splits the space $\mathbb{C} \otimes C\ell^+(\eta)$ into two orthogonal four-dimensional subspaces. The remaining freedom may be accounted for by $SU(2) \otimes SU(2) \subset SO(4)$. We form the infinitesimal generators

$$J_k^\pm = \tfrac{1}{2}\left(T_{0k} \pm \tfrac{1}{2}\epsilon_k{}^{lm} T_{lm}\right) \tag{28}$$

where each triplet, $(J_k^+)_k$ and $(J_k^-)_k$, generates a copy of $SU(2)$. We can project out isospin states about a common axis $\mathbf{a} = a^k \mathbf{e}_k$ using the commuting orthogonal pairs of projections,

$$\Lambda_\pm^+ = \tfrac{1}{2}\left(1 \pm \mathrm{i} a^k J_k^+\right) \tag{29}$$
$$\Lambda_\pm^- = \tfrac{1}{2}\left(1 \pm \mathrm{i} a^k J_k^-\right) \tag{30}$$

satisfying $\Lambda_+^{\pm} + \Lambda_-^{\pm} = 1$. Thus we can decompose $\psi \in \mathbb{C} \otimes C\ell^+(\eta)$ according to $\psi = \psi_{+++} + \psi_{++-} + \cdots + \psi_{---}$, where $\psi_{+++} = S_+\psi\Lambda_+^+\Lambda_+^-$, $\psi_{++-} = S_+\psi\Lambda_+^+\Lambda_-^-$, $\ldots$, $\psi_{---} = S_-\psi\Lambda_-^+\Lambda_-^-$ with $S_\pm = \frac{1}{2}\left(1 \pm \mathrm{i}\frac{1}{2}\epsilon_k{}^{lm}b^k\mathbf{e}_{lm}\right)$ being the projection about a spin axis $\mathbf{b} = b^k\mathbf{e}_k$ such that $\mathbf{a} \wedge \mathbf{b} \neq 0$. In particular, on $\mathbb{C} \otimes C\ell^+(\eta)$ the electroweak gauge group $U(1) \otimes SU(2)$ is admitted. This may be used to formulate the Weinberg–Salam electroweak theory [17, 18].

3. Extracting $U(1) \otimes SU(3) \subset SO(6)$

In this section we determine generators for $U(1) \otimes SU(3)$, commuting with the $U(1)$ phase generator. This is mathematical and follows Georgi [19]. We extract $SU(3)$ as a subgroup of $SO(6)$ by embedding it in the real Clifford algebra $C\ell(6, 0)$. Let $\mathbf{f}_0, \ldots, \mathbf{f}_5$ generate $C\ell(6, 0)$ through the conditions $\mathbf{f}_\mu^2 = 1$, $\mathbf{f}_\mu^\dagger = \mathbf{f}_\mu$ and $\mathbf{f}_\mu\mathbf{f}_\nu = -\mathbf{f}_\nu\mathbf{f}_\mu$ whenever $\mu \neq \nu$. The (real) subspace spanned by $\mathbf{f}_{\mu\nu}$ is a Lie algebra isomorphic to $SO(6)$ under the identification

$$\mathbf{f}_{01} \mapsto T_0 \tag{31}$$
$$\mathbf{f}_{02} \mapsto T_1 \tag{32}$$
$$\mathbf{f}_{03} \mapsto T_2 \tag{33}$$
$$\mathbf{f}_{04} \mapsto T_3 \tag{34}$$
$$\mathbf{f}_{05} \mapsto T_{0123}. \tag{35}$$

Complexifying we form the operators

$$\mathbf{A}_1 = \tfrac{1}{2}(\mathbf{f}_0 + \mathrm{i}\mathbf{f}_1) \tag{36}$$
$$\mathbf{A}_2 = \tfrac{1}{2}(\mathbf{f}_2 + \mathrm{i}\mathbf{f}_3) \tag{37}$$
$$\mathbf{A}_3 = \tfrac{1}{2}(\mathbf{f}_4 + \mathrm{i}\mathbf{f}_5). \tag{38}$$

These operators satisfy $\mathbf{A}_i \cdot \mathbf{A}_j = 0$, $\mathbf{A}_i^\dagger \cdot \mathbf{A}_j^\dagger = 0$ and $\mathbf{A}_i \cdot \mathbf{A}_j^\dagger = \delta_{ij}$. We construct generators for $SU(3)$ from the Gell-Mann matrices λ^a according to

$$\mathbf{T}^a = \sum_{ij} \mathbf{A}_i^\dagger(\lambda^a)_{ij}\mathbf{A}_j. \tag{39}$$

Since $\mathrm{Tr}\,\lambda^a = 0$, then this formula reduces to

$$\mathbf{T}^a = \sum_{ij} (\lambda^a)_{ij}[\mathbf{A}_i^\dagger, \mathbf{A}_j]. \tag{40}$$

Reading off the coefficients of the Gell-Mann matrices gives the generators for $SU(3)$ as

$$\mathbf{T}^1 = \tfrac{1}{2}(\mathbf{f}_{12} - \mathbf{f}_{03}) \tag{41}$$
$$\mathbf{T}^2 = \tfrac{1}{2}(\mathbf{f}_{31} - \mathbf{f}_{02}) \tag{42}$$
$$\mathbf{T}^3 = \tfrac{1}{2}(\mathbf{f}_{23} - \mathbf{f}_{01}) \tag{43}$$
$$\mathbf{T}^4 = \tfrac{1}{2}(\mathbf{f}_{14} - \mathbf{f}_{05}) \tag{44}$$
$$\mathbf{T}^5 = \tfrac{1}{2}(\mathbf{f}_{04} + \mathbf{f}_{15}) \tag{45}$$
$$\mathbf{T}^6 = \tfrac{1}{2}(\mathbf{f}_{34} - \mathbf{f}_{25}) \tag{46}$$
$$\mathbf{T}^7 = \tfrac{1}{2}(\mathbf{f}_{24} + \mathbf{f}_{35}) \tag{47}$$
$$\mathbf{T}^8 = \tfrac{1}{2\sqrt{3}}(2\mathbf{f}_{45} - \mathbf{f}_{01} - \mathbf{f}_{23}). \tag{48}$$

A generator for $U(1)$ commuting with generators for $SU(3)$ is given by

$$\mathbf{S} = \mathrm{i}\sum_i \mathbf{A}_i^\dagger \mathbf{A}_i - \frac{3\mathrm{i}}{2} = \mathrm{i}\sum_i [\mathbf{A}_i^\dagger, \mathbf{A}_i] \tag{49}$$

$$= -\tfrac{1}{2}(\mathbf{f}_{01} + \mathbf{f}_{23} + \mathbf{f}_{45}). \tag{50}$$

This shows that $U(1) \otimes SU(3) \subset SO(6)$.

4. $SU(3)$ colour in spacetime algebra

Transcribing the $SU(3)$ generators in terms of T_α, using (31) to (35), gives

$$T^1 = \tfrac{1}{2}(-T_{01} - T_2) = -\tfrac{1}{2}T_2(1 - \mathrm{i}T_{012}) \tag{51}$$
$$T^2 = \tfrac{1}{2}(T_{02} - T_1) = -\tfrac{1}{2}T_1(1 - \mathrm{i}T_{012}) \tag{52}$$
$$T^3 = \tfrac{1}{2}(-T_{12} - T_0) = -\tfrac{1}{2}T_{12}(1 - \mathrm{i}T_{012}) \tag{53}$$
$$T^4 = \tfrac{1}{2}(-T_{03} - T_{0123}) = -\tfrac{1}{2}T_{03}(1 + \mathrm{i}T_{12}) \tag{54}$$
$$T^5 = \tfrac{1}{2}(T_3 + T_{123}) = \tfrac{1}{2}T_3(1 + \mathrm{i}T_{12}) \tag{55}$$
$$T^6 = \tfrac{1}{2}(-T_{23} + T_{023}) = -\tfrac{1}{2}T_{23}(1 - \mathrm{i}T_0) \tag{56}$$
$$T^7 = \tfrac{1}{2}(T_{31} - T_{031}) = \tfrac{1}{2}T_{31}(1 - \mathrm{i}T_0) \tag{57}$$
$$T^8 = \tfrac{1}{2\sqrt{3}}(-2T_{012} - T_0 + T_{12})$$
$$= \tfrac{1}{\sqrt{3}}\left(\tfrac{1}{2}T_{12}(1 - \mathrm{i}T_0) - \tfrac{1}{2}T_0(1 + \mathrm{i}T_{12})\right). \tag{58}$$

We note that $T^1T^2 = T^3$ motivating us to define the dependent generators $T^9 = T^4T^5$ and $T^{10} = T^6T^7$. Explicitly these are given by

$$T^9 = \tfrac{1}{2}T_0(1 + \mathrm{i}T_{12}) \tag{59}$$
$$T^{10} = -\tfrac{1}{2}T_{12}(1 - \mathrm{i}T_0). \tag{60}$$

Thus $T^8 = -\frac{1}{\sqrt{3}}(T^{10} + T^9)$. Moreover, the generators are of the form (up to a sign) $T^a = \frac{1}{2}T_\alpha(1 \pm \mathrm{i}T_\beta)$ where $[T_\alpha, T_\beta] = 0$ and $a = 1, 2, \ldots, 10$. Since ${T_\alpha}^2 = -1$, then $(T^a)^2 = -\frac{1}{2}(1 \pm \mathrm{i}T_\beta)$ which is a projection operator. We define from the squares of T^3, T^9 and T^{10} the inter-dependent right acting orthogonal projection pairs:

$$S_\pm = \tfrac{1}{2}(1 \mp \mathrm{i}T_{012}) \tag{61}$$
$$T_\pm = \tfrac{1}{2}(1 \pm \mathrm{i}T_{12}) \tag{62}$$
$$R_\pm = \tfrac{1}{2}(1 \mp \mathrm{i}T_0). \tag{63}$$

These projections satisfy

$$R_+T_+ = S_+T_+ = R_+S_+ \tag{64}$$
$$R_-T_- = S_+T_- = R_-S_+ \tag{65}$$
$$R_-T_+ = S_-T_+ = R_-S_- \tag{66}$$
$$R_+T_- = S_-T_- = R_+S_-. \tag{67}$$

Table 1. Eigenvalues of T^3, T^9 and T^{10} operators.

	$\mathrm{i}T^3$	$\mathrm{i}T^9$	$\mathrm{i}T^{10}$	$\sqrt{3}\mathrm{i}T^8$
S_+T_+	1	1	1	−2
S_+T_-	−1	0	0	0
S_-T_+	0	−1	0	1
S_-T_-	0	0	−1	1

Table 2. Eigenvalues of isospin (T) and hypercharge (Y).

	T	Y
S_+T_+	0	$-\frac{2}{3}$
S_+T_-	0	0
S_-T_+	$\frac{1}{2}$	$\frac{1}{3}$
S_-T_-	$-\frac{1}{2}$	$\frac{1}{3}$

It follows that $R_\pm = S_+T_\pm + S_-T_\mp$ and consequently that

$$S_+T_+R_+ = S_+T_+ \qquad S_+T_+R_- = 0 \tag{68}$$

$$S_+T_-R_+ = 0 \qquad S_+T_-R_- = S_+T_- \tag{69}$$

$$S_-T_+R_+ = 0 \qquad S_-T_+R_- = S_-T_+ \tag{70}$$

$$S_-T_-R_+ = S_-T_- \qquad S_-T_-R_- = 0. \tag{71}$$

We may decompose any spinor $\psi \in \mathbb{C} \otimes C\ell(\eta)$ according to the four-dimensional subalgebras $\mathbb{C} \otimes C\ell(\eta)S_\pm T_\pm$. Write $\psi = \psi_0 - \mathrm{i}\sqrt{\eta}\psi_1\mathbf{e}_0$, where $\psi_0, \psi_1 \in \mathbb{C} \otimes C\ell^+(\eta)$. Since $\mathbf{e}_0 R_\pm = \pm\mathrm{i}\eta\sqrt{\eta}R_\pm$ then $\psi R_\pm = (\psi_0 \pm \psi_1)R_\pm$. Thus $\psi = \psi_+R_+ + \psi_-R_-$, or equivalently

$$\psi = \psi_+(S_+T_+ + S_-T_-) + \psi_-(S_+T_- + S_-T_+) \tag{72}$$

where $\psi_\pm = \psi_0 \pm \psi_1 \in \mathbb{C} \otimes C\ell^+(\eta)$. Each component lies in a four-dimensional subalgebra $\mathbb{C} \otimes C\ell(\eta)S_\pm T_\pm$. Since $T_{12}T_\pm = \mp\mathrm{i}T_\pm$ and $T_0R_\pm = \pm\mathrm{i}R_\pm$ we may deduce that the eigenvalues are given by table 1. We obtain the usual $SU(3)$ eigenvalues by defining the isospin operator to be

$$T = \tfrac{1}{2}\mathrm{i}(-T^9 + T^{10}) \tag{73}$$

and the hypercharge operator to be

$$Y = \tfrac{1}{\sqrt{3}}\mathrm{i}T^8. \tag{74}$$

Now the eigenvalues labelling the four projection operators $S_\pm T_\pm$ are given in table 2. Raising and lowering operators are given by

$$U^\pm = -\tfrac{1}{2}(T^1 \mp \mathrm{i}T^2) \tag{75}$$

$$V^\pm = \tfrac{1}{2}(T^5 \pm \mathrm{i}T^4) \tag{76}$$

$$W^\pm = \tfrac{1}{2}(T^6 \mp \mathrm{i}T^7). \tag{77}$$

These satisfy

$$S_-T_\mp U^\pm = T_2 S_-T_\pm \tag{78}$$

$$S_\pm T_+ V^\pm = T_3 S_\mp T_+ \tag{79}$$

$$S_\pm T_\pm W^\pm = T_{23} S_\mp T_\mp \tag{80}$$

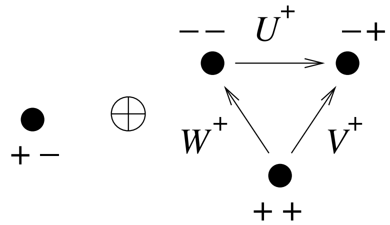

Figure 2. Direct sum decomposition of $\mathbb{C}\otimes C\ell(\eta)$ under $SU(3)$ into four-dimensional left ideals.

with all other combinations annihilated. Thus we see that the space $\mathbb{C} \otimes C\ell(\eta)$ decomposes into four four-dimensional subalgebras given by the direct sum $[1] \oplus [3]$. This is depicted in figure 2. The spinor field ψ is a spin half field and so this $SU(3)$ gauge freedom cannot be associated with flavour. We propose that it is coloured and identify the colourless [1] subalgebra with a lepton field and the [3] subalgebra with a quark field. Thus every ψ field is composed of a colourless leptonic field and a coloured quark field. This mirrors the experimental fact that there are equal numbers of quark and lepton flavours.

5. The colour and electromagnetic interactions

The gauge freedom uncovered, in addition to a $U(1)$ freedom for phase, provides a $U(1) \otimes SU(3)$ global gauge right action on a Dirac field. This field decomposes under the $SU(3)$ right action into

$$\psi = \psi_l + \psi_q \tag{81}$$

where $\psi_l = \psi S_+T_-$ is the leptonic component and $\psi_q = \psi(1 - S_+T_-)$ the quarkonic component. The quarkonic component splits into the three colours,

$$\psi_q = \psi_r + \psi_g + \psi_b \tag{82}$$

where the red, green and blue quark colour components are

$$\psi_r = \psi S_-T_+ \tag{83}$$

$$\psi_g = \psi S_-T_- \tag{84}$$

$$\psi_b = \psi S_+T_+. \tag{85}$$

The conjugate of the quarkonic field provides us with an anti-quarkonic field with colour components anti-red, anti-green and anti-blue.

The $U(1)$ generator $S = -\frac{1}{2}(T_0 - T_{12} - T_{012})$ from (50) defines a charge operator $Q = -\mathrm{i}\frac{2}{3}\mathbf{S}$ which may be written in the form

$$Q = -\mathrm{i}\tfrac{1}{3}(2T_{12}S_- + T_{012}). \tag{86}$$

Now it is straightforward to verify, by applying Q to $S_\pm T_\pm$ and noting that $T_{12}T_\pm = \mp\mathrm{i}T_\pm$ and $T_{012}S_\pm = \pm\mathrm{i}S_\pm$, that the components of ψ have the charges given by table 3. Moreover, the charge decomposes into the sum of a quarkonic and a leptonic charge operator

$$Q = \tfrac{1}{3}(1 - S_-T_+) - S_-T_+. \tag{87}$$

Minimal local gauging requires the introduction of gauge fields. Two $U(1)$ gauge fields for the generators i and $\boldsymbol{\lambda} = \frac{2}{3}\mathbf{S}$, and gluon fields for $SU(3)$. The phase $U(1)$ gauge field is denoted by $\mathbf{A} = A^\mu\mathbf{e}_\mu$ and the non-phase $U(1)$ gauge field by $\mathbf{B} = B^\mu\mathbf{e}_\mu$. The eight gluon generators transfer colour between quarks and are defined in table 4. The gluon fields

Table 3. Charge assignments to the components of the Dirac field.

Component	Q
ψ_l	-1
ψ_r	$\frac{1}{3}$
ψ_g	$\frac{1}{3}$
ψ_b	$\frac{1}{3}$

Table 4. Gluon generators and the colour transferred.

Gluon	Generator	Colour change
$\boldsymbol{\lambda}^1$	U^+	$g\bar{r}$
$\boldsymbol{\lambda}^2$	U^-	$r\bar{g}$
$\boldsymbol{\lambda}^3$	V^+	$b\bar{r}$
$\boldsymbol{\lambda}^4$	V^-	$r\bar{b}$
$\boldsymbol{\lambda}^5$	W^+	$b\bar{g}$
$\boldsymbol{\lambda}^6$	W^-	$g\bar{b}$
$\boldsymbol{\lambda}^7$	$\mathrm{i}T$	$\frac{r\bar{r}-g\bar{g}}{2}$
$\boldsymbol{\lambda}^8$	$\mathrm{i}\sqrt{3}Y$	$\frac{2b\bar{b}-r\bar{r}-g\bar{g}}{3}$

Table 5. Quantum numbers of the Dirac component fields.

Colour	T	Y	Q
Black	0	0	-1
r	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$
g	$-\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$
b	0	$-\frac{2}{3}$	$\frac{1}{3}$

are given by $\mathbf{G}^a = G^\mu_a \mathbf{e}_\mu$, where $a = 1, \ldots, 8$. The non-phase $U(1)$ gauge field has (in a sense) a gluonic component $\frac{1}{3}(r\bar{r} + g\bar{g} + b\bar{b})$. The quantum number assignment of the colour components (including the leptonic component) is given in table 5. The lepton is viewed as having, by convention, no colour (or is black). Note that every column sums to zero.

The covariant derivative is given by

$$D^\mu \psi = \partial^\mu \psi - \frac{g'}{2} A^\mu \psi + \frac{g}{2} B^\mu \psi \mathrm{i}\boldsymbol{\lambda} + \frac{G}{2} G^\mu_a \psi \mathrm{i}\lambda^a \tag{88}$$

where g, g' are the coupling coefficients for the $U(1)$ gauge fields, and G is the coupling coefficient for the gluon fields. We mix the $U(1)$ gauge fields by defining new fields

$$\begin{pmatrix} X^\mu \\ Y^\mu \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} A^\mu \\ B^\mu \end{pmatrix} \tag{89}$$

where the mixing angle is given by

$$\cos\theta = \frac{g'}{\sqrt{g^2 + g'^2}} \tag{90}$$

and the new coupling coefficient is given by

$$q = \frac{gg'}{\sqrt{g^2 + g'^2}} \tag{91}$$

$$= \tfrac{1}{2}\sin 2\theta. \tag{92}$$

Table 6. Charge assignments of the Dirac component fields.

Flavour	Charge projection	Charge	Particles
l	Q_+	-1	e, μ, τ
l	Q_-	0	ν_e, ν_μ, ν_τ
q	Q_+	$-\frac{1}{3}$	u, s, t
q	Q_-	$\frac{2}{3}$	d, c, b

The $\mathbf{X} = X^\mu \mathbf{e}_\mu$ is associated with $U(1)$ gauge group generated by $\mathrm{i}\frac{1}{2}(1 + \mathrm{i}\boldsymbol{\lambda})$. Similarly, $\mathbf{Y} = Y^\mu \mathbf{e}_\mu$ is associated with the generator $\mathrm{i}\frac{1}{2}(1 - \mathrm{i}\boldsymbol{\lambda})$. The covariant derivative is now given by

$$D^\mu \psi = \partial^\mu \psi - \frac{q}{2} X^\mu \psi \frac{1}{2}(1 + \mathrm{i}\boldsymbol{\lambda}) - \frac{q}{2} B^\mu \psi \frac{1}{2}(1 - \mathrm{i}\boldsymbol{\lambda}) + \frac{G}{2} G^\mu_a \psi \mathrm{i}\lambda^a. \tag{93}$$

The charge (projection) operators $Q_\pm = \frac{1}{2}(1 \pm \mathrm{i}\lambda)$ do not contribute another degree of freedom. There are only two available and these are already taken by the colour projections. Nevertheless the charge assignments match those required by a generation of the standard model and are given in table 6.

We may break $U(1)$ symmetry by including a Lagrange multiple of either of the $U(1)$ gauge fields into the Lagrangian. This annihilates this gauge field, thus breaking the gauge. Breaking $U(1)$ symmetry corresponding to Q_- (respectively Q_+) gives an electro-colour Lagrangian corresponding to a lepton flavour from row 1 (respectively 2) and a quark flavour from row 3 (respectively 4) of table 6. We require another degree of freedom in $\mathbb{C} \otimes C\ell(\eta)$ in order to have isospin doublets. This would presumably require an incorporation of $SU(2)$ isospin.

6. Conclusion

We analysed the Dirac equation of Joyce [14] over complexified spacetime algebra $\mathbb{C} \otimes C\ell(\eta)$. We established a maximal gauge group acting faithfully and independently of the Lorentz group. This was found to be $U(1) \otimes U(1) \otimes SU(3)$. We explicitly determined generators for the subgroup which we used to decompose the Dirac field ψ into a colourless leptonic part (transforming as [1]) and a coloured quarkonic part (transforming as [3]). This together with up/down spin and particle/anti-particle eigenstates, accounts for the four degrees of freedom. Moreover, this indicates a possible origin for the $SU(3)$ colour gauge freedom. It also explains the observation that there are equal numbers of lepton and quark flavours. The gauge fields corresponding to $U(1) \otimes U(1)$ mix to give photon fields, one with (lepton, quark) quantum numbers $\left(1, -\frac{1}{3}\right)$ and the other with $\left(0, \frac{2}{3}\right)$. Unfortunately, the complexified spacetime algebra requires another degree of freedom to project out a complete generation of the standard model. This is the only obstacle to an electro-colour description of a single generation of the standard model in $\mathbb{C} \otimes C\ell(\eta)$.

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